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On Time Dynamics of Coagulation-Fragmentation Processes

Boris L. Granovsky · Michael M. Erlihson

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Abstract We establish a characterization of coagulation-fragmentation processes, such that the induced birth and death processes depicting the total number of groups at time $t \geq 0$ are time homogeneous. Based on this, we provide a characterization of mean-field Gibbs coagulation-fragmentation models, which extends the one derived by Hendriks et al. As a by-product of our results, the class of solvable models is widened and a question posed by N. Berestycki and Pitman is answered, under restriction to mean-field models.

Keywords Time dynamics · Coagulation-fragmentation models · Gibbs distributions on the set of integer partitions

1 Introduction, Objective and the Context

The time dynamics of a time homogeneous Markov process $X(t),\ t \geq 0$ on a space $\Omega = \{\eta\}$ of states η is described by the set of transition probabilities

$$p_{\tilde{\zeta}}(\eta; t) := \mathbb{P}(X(t) = \eta \mid X(0) = \tilde{\zeta}), \quad \tilde{\zeta}, \eta \in \Omega,\ t \geq 0.$$

Given the rates of the infinitesimal state transitions, the explicit expressions for the transition probabilities $p_{\tilde{\zeta}}$ as solutions of a Kolmogorov system, are known only for a few special cases of the rates. The corresponding models are called solvable. For the above reason, time dynamics of Markov processes remain, generally speaking, a mystery. As an example, even for birth-death processes on the set of integers, the explicit solutions have been derived only for a few combinations of birth and death rates. This explains why the direction of research in this area turned to the estimation of the rate of convergence (=spectral gap) of

B.L. Granovsky (✉) · M.M. Erlihson
Department of Mathematics, Technion-Israel Institute of Technology, Haifa, 32000, Israel
e-mail: mar18aa@techunix.technion.ac.il

M.M. Erlihson
e-mail: maerlich@tx.technion.ac.il

the transition probabilities as $t \to \infty$. Nevertheless, hunting for solvable models continues to be of interest.

In the present paper we pursue the above objective for stochastic processes of coagulation and fragmentation (*CFP*'s). We adopt the formulation of a $CFP = CFP(N)$ given in [5] on the basis of classic works of Whittle [25] and Kelly [14] devoted to deterministic and stochastic models of clustering in polymerization, electrical networks and in a variety of other fields. A *CFP* $X_N(t),\ t \geq 0$ is defined as a time homogeneous Markov chain on the state space Ω_N of all partitions $\eta = (n_1, \ldots, n_N) : \sum_{i=1}^N in_i = N$, of a given integer N. Here N codes the total population of indistinguishable particles partitioned into groups (=clusters) of different sizes, while n_i is the number of groups of size i. Infinitesimal (in time) events are a coagulation of two groups into one and a fragmentation of one group into two groups, and the basic assumption is that the rates (intensities) of the above two single transitions depend only on sizes of groups (and do not depend on N). Namely, the rate of a single coagulation of two groups of sizes i and j, such that $2 \leq i + j \leq N$, into one group of size $i + j$ is $\psi(i, j)$, whereas the rate of a single fragmentation of a group with size $i + j$ into two groups of sizes i and j is $\phi(i, j)$. The functions ψ and ϕ are assumed to be non negative and symmetric in i, j.

Next, we define the induced rates of infinitesimal state transitions. Given a state $\eta \in \Omega_N$ with $n_i, n_j > 0$ for some $1 \leq i, j \leq N$, denote by $\eta^{(i,j)} \in \Omega_N$ the state that is obtained from η by a coagulation of any two groups of sizes i and j, and denote by $K(\eta \to \eta^{(i,j)})$ the rate of the infinitesimal state transition $\eta \to \eta^{(i,j)}$. Similarly, for a given state $\eta \in \Omega_N$ with $n_{i+j} > 0$, let $\eta_{(i,j)}$ be the state that is obtained from η by a fragmentation of a group of size $i + j \geq 2$ into two groups of sizes i and j, and let $F(\eta \to \eta_{(i,j)})$ be the rate of the infinitesimal state transition $\eta \to \eta_{(i,j)}$. We assume that the rate $K(\eta \to \eta^{(i,j)})$ is equal to the sum of rates of all single coagulations of n_i groups with size i with n_j groups with size j, and that $F(\eta \to \eta_{(i,j)})$ is the sum of rates of all single fragmentations of n_{i+j} groups with size $i + j$ into two groups of sizes i and j. As a result, we get the following expressions for the rates of state transitions:

$$\begin{aligned} K(\eta \to \eta^{(i,j)}) &= n_i n_j \psi(i, j), \quad i \neq j,\ 2 \leq i + j \leq N, \\ K(\eta \to \eta^{(i,i)}) &= \frac{n_i(n_i - 1)}{2}\, \psi(i, i), \quad 2 \leq 2i \leq N, \\ F(\eta \to \eta_{(i,j)}) &= n_{i+j}\phi(i, j), \quad 2 \leq i + j \leq N. \end{aligned} \tag{1.1}$$

We note that an interpretation of the coagulation kernel K in terms of the kinetics of droplets of different masses can be found in [19].

Following [11], we call *CFP*'s with rates of state transitions of the form (1.1) mean field models, meaning that at any state $\eta \in \Omega_N$, any group can coagulate with any other one or can be fragmented into any two parts. We also note that a characterization of positive rates of single transitions $\psi(i, j), \phi(i, j)$ that provide reversibility of mean-field *CFP*'s is known ([5]).

We now describe briefly the context of the present paper. The paper is devoted to the time evolution of the above mean field *CFP*'s and it consists of two sections. Section 2 is divided into three subsections. In Sect. 2.1 we characterize the *CFP*'s $X_N(t),\ t \geq 0$ having time homogeneous processes $|X_N(t)|,\ t \geq 0$ depicting the total number of groups at time $t \geq 0$. The key result of the paper, stated precisely in Theorem 1 in Sect. 2.2, establishes the equivalence of the following two conditions:

(i) The birth and death process $|X_N(t)|,\ t \geq 0$ is time homogeneous;

(ii) The conditional distribution of a $CFP(N)$, given a total number of groups at time $t \geq 0$, is a time independent Gibbs distribution on the set of partitions of N with given number of components.

Consequently, a characterization of Gibbs CFP's, which extends the one by Hendriks et al. [13], is derived.

In Sect. 2.3 we discuss the following three topics related to our main result: Steady state distributions of CFP's, Gibbs CFP's on set partitions and Spectral gaps of Gibbs CFP's. In particular, under restriction to mean-field models, we obtain a negative answer to a question posed by N. Berestycki and Pitman [4] about the existence of certain Gibbs CFP's.

2 Main Result

We say that states $\eta, \tilde{\eta} \in \Omega_N$ are neighbors: $\tilde{\eta} \sim \eta$, if one of the states is obtained either by a single coagulation or a single fragmentation of components of the other state. Then the preceding description of a $CFP(N)$, say $X_N^{(\rho)}(t),\ t \geq 0$, starting from an initial probability distribution ρ on Ω_N allows us to write the corresponding Kolmogorov system as follows

$$\dot{p}_\rho(\eta; t) = -p_\rho(\eta; t)\Bigg(\sum_{\tilde{\eta}\sim\eta}\big(K(\eta \to \tilde{\eta}) + F(\eta \to \tilde{\eta})\big)\Bigg)$$

$$+ \sum_{\tilde{\eta}\sim\eta} p_\rho(\tilde{\eta}; t)\big(K(\tilde{\eta} \to \eta) + F(\tilde{\eta} \to \eta)\big), \quad \tilde{\eta}, \eta \in \Omega_N,\ t \geq 0. \qquad (2.1)$$

Note that the seminal system of Smoluchowski equations (1918) for pure coagulation can be viewed as an approximation to (2.1) obtained by neglecting correlations between group numbers at time $t \geq 0$. This issue is widely discussed in the literature (see [1, 5, 8, 20]).

2.1 Process of the Total Number of Groups

In our study of time dynamics of a CFP $X_N^{(\rho)}(t) = (n_1(t), \dots, n_N(t)) \in \Omega_N,\ t \geq 0$, a central role is played by the induced stochastic process

$$|X_N^{(\rho)}(t)| := \sum_{i=1}^N n_i(t), \quad t \geq 0, \qquad (2.2)$$

which depicts the total number of groups in the generic CFP at time $t \geq 0$. We denote throughout the paper

$$\Omega_{N,r} = \{\eta \in \Omega_N : |\eta| = r\}, \quad r = 1, \dots, N$$

the set of all partitions of N with exactly r components.

It follows from the definition of a $CFP(N)$ that $|X_N^{(\rho)}(t)|,\ t \geq 0$ is a Markov birth and death process on the state space $\{1, 2, \dots, N\}$, with rates of birth and death $\lambda_{r,N},\ 1 \leq r \leq N-1,\ \mu_{r,N},\ 2 \leq r \leq N$, respectively, defined in a usual way, as in (2.3) below. However, in contrast to the generic $CFP(N)$, the process $|X_N^{(\rho)}(t)|,\ t \geq 0$ is, in general, not homogeneous in time, which presents a big problem for the investigation of the process.

The following example demonstrates the phenomenon of dependence of the rates $\lambda_{r,N}, \mu_{r,N}$ on time $t \geq 0$ and on an initial distribution ρ of the generic $CFP(N)$, that causes the time-inhomogeneity of the induced birth and death process.

Example 1 Consider a $CFP(N)$, $N > 4$ of pure coagulation with $\psi(1,1) = \psi(1,2) = 0$ and all other $\psi(i,j) > 0$. It is clear that $\Omega_{N,N-2} = \eta_1 \cup \eta_2$, where $\eta_1 = (N-3,0,1,0,\dots,0)$, $\eta_2 = (N-4,2,0,\dots,0)$. Assuming that the process starts from an initial distribution ρ on $\Omega_{N,N-2}$, s.t. $\rho(\eta_i) = p_i > 0,\ i = 1,2,\ p_1 + p_2 = 1$ and denoting $A_i = \sum_{\zeta\in\Omega_{N,N-3}} K(\eta_i \to \zeta) > 0,\ i = 1,2$, we have

$$\dot{p}_\rho(\eta_i;t) = -A_i p_\rho(\eta_i;t), \quad t \ge 0,\ i = 1,2,$$

since the transitions $\Omega_{N,N-1} \to \eta_i,\ i = 1,2$ are impossible. Hence,

$$p_\rho(\eta_i;t) = p_i e^{-A_i t}, \quad t \ge 0,\ i = 1,2$$

and consequently, it follows from (2.7) below that the rate of death

$$\mu_{N-2,N}(t;\rho) = \frac{A_1 e^{-A_1 t} p_1 + A_2 e^{-A_2 t} p_2}{e^{-A_1 t} p_1 + e^{-A_2 t} p_2}, \quad t \ge 0$$

depends on ρ and t iff $A_1 = (N-3)\psi(1,3) \ne A_2 = \psi(2,2)$.

We further assume that the *CFP*'s considered are ergodic.

We first distinguish *CFP*'s(N) which induce time homogeneous processes $|X_N^{(\rho)}(t)|$, $t \ge 0$, under any initial distribution ρ on Ω_N, i.e. processes with birth and death rates not depending on $t \ge 0$ and ρ. Let $\lambda_{r,N}(t;\rho),\ 1 \le r \le N-1$ and $\mu_{r,N}(t;\rho),\ 2 \le r \le N$ be, respectively, the rates of birth and death at time $t \ge 0$, of some $|X_N^{(\rho)}(t)|$:

$$\begin{aligned} \lambda_{r,N}(t;\rho) &= \lim_{\Delta t \to 0^+} \frac{\mathbf{P}(|X_N^{(\rho)}(t+\Delta t)| = r+1 \mid |X_N^{(\rho)}(t)| = r)}{\Delta t}, \\ \mu_{r,N}(t;\rho) &= \lim_{\Delta t \to 0^+} \frac{\mathbf{P}(|X_N^{(\rho)}(t+\Delta t)| = r-1 \mid |X_N^{(\rho)}(t)| = r)}{\Delta t}. \end{aligned} \tag{2.3}$$

Equation (2.3) tells us that under the assumption of ergodicity of the generic $CFP(N)$, the time independence of birth and death rates implies their independence on ρ.

Clearly, the birth and death rates in (2.3) are implied respectively, by the rates ψ of single fragmentations and the rates ϕ of single coagulations of the generic $CFP(N)$. It turns out that the required necessary and sufficient condition of time homogeneity of the process $|X_N^{(\rho)}(t)|,\ t \ge 0$ has a simple probabilistic meaning.

Lemma 1 $|X_N^{(\rho)}(t)|,\ t \ge 0$ *is a time homogeneous birth and death process under any initial distribution* ρ *on* Ω_N, *if and only if the generic* $CFP(N)$ *is such that for a given* $1 \le r \le N$ *the sums of rates* $\sum_{\tilde\eta\sim\eta} K(\eta \to \tilde\eta),\ \eta \in \Omega_{N,r}$ *and* $\sum_{\tilde\eta\sim\eta} F(\eta \to \tilde\eta),\ \eta \in \Omega_{N,r}$ *do not depend on* $\eta \in \Omega_{N,r}$. *Under the above condition, the first sum and the second sum are equal to the rate of death* $\mu_{r,N}$ *and to the rate of birth* $\lambda_{r,N}$ *respectively, so that for any* $\eta \in \Omega_{N,r}$,

$$\begin{aligned} \lambda_{r,N} &= \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbf{P}\Big(|X_N^{(\rho)}(t+\Delta t)| = r+1 \mid X_N^{(\rho)}(t) = \eta\Big), \quad 1 \le r \le N-1, \\ \mu_{r,N} &= \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbf{P}\Big(|X_N^{(\rho)}(t+\Delta t)| = r-1 \mid X_N^{(\rho)}(t) = \eta\Big), \quad 2 \le r \le N, \end{aligned} \tag{2.4}$$

under any initial distribution ρ *on* Ω_N *and all* $t \ge 0$.

Proof Recalling that $|X_N^{(\rho)}(t)|,\ t\ge 0$ is Markov, by the Markovian property of the generic $CFP(N)$, we firstly assume that $|X_N^{(\rho)}(t)|,\ t\ge 0$ is time homogeneous, so that for all ρ on Ω_N and all $t\ge 0$, $\lambda_{r,N}(t;\rho)=\lambda_{r,N},\ 1\le r\le N-1$ and $\mu_{r,N}(t;\rho)=\mu_{r,N},\ 2\le r\le N$. We now rewrite (2.3) as

$$\begin{aligned}
\lambda_{r,N} &= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\frac{\sum_{\eta\in\Omega_{N,r}}\mathbf{P}(|X_N^{(\rho)}(t+\Delta t)|=r+1\mid X_N^{(\rho)}(t)=\eta)\mathbf{P}(X_N^{(\rho)}(t)=\eta)}{\mathbf{P}(|X_N^{(\rho)}(t)|=r)},\\
&\quad 1\le r\le N-1,\\
\mu_{r,N} &= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\frac{\sum_{\eta\in\Omega_{N,r}}\mathbf{P}(|X_N^{(\rho)}(t+\Delta t)|=r-1\mid X_N^{(\rho)}(t)=\eta)\mathbf{P}(X_N^{(\rho)}(t)=\eta)}{\mathbf{P}(|X_N^{(\rho)}(t)|=r)},\\
&\quad 2\le r\le N.
\end{aligned}\tag{2.5}$$

By the ergodicity and the time homogeneity properties of the generic $CFP(N)$, the limits

$$\begin{aligned}
f_b(\eta;r,N) &:= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\mathbf{P}\Big(|X_N^{(\rho)}(t+\Delta t)|=r+1\mid X_N^{(\rho)}(t)=\eta\Big),\quad 1\le r\le N-1,\\
f_d(\eta;r,N) &:= \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\mathbf{P}\Big(|X_N^{(\rho)}(t+\Delta t)|=r-1\mid X_N^{(\rho)}(t)=\eta\Big),\quad 2\le r\le N,
\end{aligned}\tag{2.6}$$

do not depend on $t\ge 0$ and ρ, for all $\eta\in\Omega_{N,r}$, so that

$$\begin{aligned}
\lambda_{r,N} &= \sum_{\eta\in\Omega_{N,r}}\frac{f_b(\eta;r,N)\mathbf{P}(X_N^{(\rho)}(t)=\eta)}{\mathbf{P}(|X_N^{(\rho)}(t)|=r)},\quad 1\le r\le N-1,\\
\mu_{r,N} &= \sum_{\eta\in\Omega_{N,r}}\frac{f_d(\eta;r,N)\mathbf{P}(X_N^{(\rho)}(t)=\eta)}{\mathbf{P}(|X_N^{(\rho)}(t)|=r)},\quad 2\le r\le N.
\end{aligned}\tag{2.7}$$

Next, setting in (2.7), $t=0$ and $\rho(\tilde\zeta)=1$, for a $\tilde\zeta\in\Omega_{N,r}$, (so that $\mathbf{P}(X_N^{(\rho)}(0)=\tilde\zeta)=1$) it is easy to conclude that (2.7) together with the time homogeneity assumption imply

$$\lambda_{r,N}=f_b(\tilde\zeta;r,N)=const,\qquad \mu_{r,N}=f_d(\tilde\zeta;r,N)=const,$$

for all $\tilde\zeta\in\Omega_{N,r}$, which proves the necessity of the condition (2.4). The sufficiency of (2.4) follows immediately from (2.7), after we observe that in view of (2.6), the quantities $f_d(\eta;r,N),\ f_b(\eta;r,N)$ are equal respectively, to the sum of rates of single coagulations $\sum_{\tilde\eta\sim\eta}K(\eta\to\tilde\eta)$ and to the sum of rates of single fragmentations $\sum_{\tilde\eta\sim\eta}F(\eta\to\tilde\eta)$ at a state $\eta\in\Omega_{N,r}$. □

In the rest of this subseqtion we will treat the case when the time homogeneity condition in Lemma 1 holds, writing simply $|X_N(t)|,\ t\ge 0$. Now our objective will be to characterize the rates $\psi(i,j),\phi(i,j)$ that provide the condition (2.4). The condition (2.4) says that for given N and r each one of the two limits in the RHS of (2.4) is the same for all $\eta\in\Omega_{N,r}$ and all ρ on Ω_N. Consequently, the above condition conforms to two separate systems of linear equations, one for $\psi(i,j)$ and one for $\phi(i,j)$, and each one consisting of $|\Omega_{N,r}|$ equations for each $1\le r\le N$. It is easily seen that for a fixed N there is a variety of solutions to each of these systems, which are valid for all $1\le r\le N$.

For example, employing the aforementioned meaning of the limits f_b and f_d, one can verify that for a given $N > 3$ the following rates depending on N satisfy (2.4):

$$\psi(i,j) = \begin{cases} i+j, & \text{if } 2 \le i+j \le N-1, \\ l_1(N), & \text{if } i+j = N \end{cases} \tag{2.8}$$

and

$$\phi(i,j) = \begin{cases} 0, & \text{if } 2 \le i+j < N, \\ l_2(N), & \text{if } i+j = N, \end{cases} \tag{2.9}$$

where l_1 and l_2 are arbitrary nonnegative functions.

However, due to our basic assumption that the rates ψ and ϕ do not depend on N, time homogeneity of the process $|X_N(t)|$ $t \ge 0$ implies a very special form of the above rates of single transitions.

Proposition 1 $\{|X_N(t)|,\ t \ge 0\}_{N\ge 1}$ *is a sequence of time homogeneous birth and death processes induced by a sequence of* $\{CFP(N)\}_{N\ge 1}$ *with rates of single transitions* $\psi(i,j)$ *and* $\phi(i,j)$, *if and only if the above rates are of the form*:

$$\psi(i,j) = a(i+j)+b, \quad i,j \ge 1,\ a \ge 0,\ 2a+b \ge 0 \tag{2.10}$$

and

$$v_k := \sum_{1\le i\le j:\ i+j=k} \phi(i,j) = \phi(1,1)(k-1), \quad k \ge 1, \tag{2.11}$$

where $v(k)$ *is the sum of rates of all possible single fragmentations of a group of size* $k \ge 2$ *into two groups*, *whereas* $v_2 = \phi(1,1) \ge 0$ *is arbitrary.*

Proof We employ the preceding lemma. Assuming that the processes $|X_N(t)|,\ t \ge 0$ are time homogeneous for all $N \ge 1$, we apply the second part of (2.4) with $r = 2$ to obtain

$$\psi(i, N-i) = \mu_{2,N}, \quad i = 1,\dots,N-1,\ N \ge 1.$$

Therefore,

$$\psi(i,j) = s(i+j), \quad i,j \ge 1, \tag{2.12}$$

where s is some nonnegative function on integers which are greater or equal to 2.

Next, consider the two states $\eta_1, \eta_2 \in \Omega_{N,3},\ N \ge 5$:

$$\begin{aligned} \eta_1 &= (2,0,\dots,0,\overbrace{\ 1\ }^{n_{N-2}},0,\dots,0), \\ \eta_2 &= (1,1,0,\dots,0,\overbrace{\ 1\ }^{n_{N-3}},0,\dots,0). \end{aligned} \tag{2.13}$$

Applying the equation $f_d(\eta_1;3,N) = f_d(\eta_2;3,N)$, gives

$$2\psi(1,N-2)+\psi(1,1) = \psi(1,N-3)+\psi(N-3,2)+\psi(1,2), \tag{2.14}$$

which by virtue of (2.12), is equivalent to

$$2s(N-1)+s(2)=s(N-2)+s(N-1)+s(3), \quad N\geq 5.$$

Taking into account that the last relation should hold for all $N \geq 5$, we rewrite it as $s(k)-s(k-1)=s(3)-s(2)$, $k\geq 3$, which proves the necessity of (2.10). For the proof of the necessity of (2.11) we consider the quantities $f_b(\eta; 2, N)$ for N fixed and all states η of the form

$$\eta=(0,\dots,0,\dots,0,\overbrace{1}^{i},0,\dots,0,\overbrace{1}^{N-i},0,\dots,0)\in\Omega_{N,2}, \quad 1\leq i\leq N-1.$$

Using the notation in (2.11), the condition that $f_b(\eta; 2, N)$ should be the same for all the above η can be written as

$$v(i)+v(N-i)=const, \quad 1\leq i\leq N-1, \tag{2.15}$$

or, equivalently, $v(N-1)-v(N-2)=v(2)-v(1)=v(2)$. Since the latter relationship should hold for all $N\geq 2$, it implies (2.11). We turn now to the proof of sufficiency of the conditions (2.10) and (2.11). Supposing that (2.10) holds, we have for a state $\eta\in\Omega_{N,r}$:

$$\begin{aligned} f_d(\eta; r, N) &= \sum_{1\leq i<j\leq N}\psi(i,j)n_in_j+\sum_{1\leq i\leq N}\psi(i,i)\frac{n_i(n_i-1)}{2} \\ &= \frac{1}{2}\Big(\sum_{1\leq i,j\leq N}\psi(i,j)n_in_j-\sum_{1\leq i\leq N}\psi(i,i)n_i\Big) \\ &= \frac{1}{2}\Big(\sum_{1\leq i,j\leq N}\big(a(i+j)+b\big)n_in_j-\sum_{1\leq i\leq N}(2ia+b)n_i\Big) \\ &= \frac{1}{2}\big(2aNr+br^2-2aN-br\big), \quad r=2,\dots,N, \\ f_b(\eta; r, N) &= \sum_{1\leq k\leq N}v(k)n_k \\ &= \sum_{1\leq k\leq N}v(2)(k-1)n_k=v(2)(N-r), \quad r=1,\dots,N-1. \end{aligned} \tag{2.16}$$

□

Corollary 1 *The rates of death and birth of a time homogeneous Markov process* $|X_N(t)|$, $t\geq 0$ *are given by*

$$\begin{aligned} \mu_{r,N} &= \frac{(r-1)}{2}(2aN+rb), \quad 2\leq r\leq N, \\ \lambda_{r,N} &= \phi(1,1)(N-r), \quad 1\leq r\leq N-1. \end{aligned} \tag{2.17}$$

Remark 1

(i) The birth and death process $|X_N(t)|$, $t\geq 0$ with rates given by (2.17), has the following interpretation, not related to the generic *CFP*(N). Consider a nearest neighbor spin system (for reference see [17]) of "0"-s and "1"-s on a complete graph on N vertices (sites). Assume that one of the sites is occupied with a "1" which never flips, while spins at all other sites perform flips $0 \to 1$ and $1 \to 0$ with rates $\tilde{\lambda}_{r,N}$ and $\tilde{\mu}_{r-1,N}$ respectively, where r is the total number of sites of the graph occupied by "1"-s. (The latter says that a site occupied by a "1" has $r-1$ neighbors occupied by 1-s and a site occupied by a "0" has r such neighbors.) Consequently, at a state with $r \geq 1$ "1"-s, the total rate of $0 \to 1$ flips is $\lambda_{r,N} := (N-r)\tilde{\lambda}_{r,N}$ and the total rate of $1 \to 0$ flips is $\mu_{r,N} := (r-1)\tilde{\mu}_{r-1,N}$. Therefore, the induced birth and death process, say $\zeta_N(t),\ t \geq 0$, on $\{1, \dots, N\}$ depicting the number of sites occupied by "1"-s at time $t \geq 0$ is Markov and time homogeneous. Clearly, if

$$\tilde{\lambda}_{r,N} = \phi(1,1), \quad 1 \leq r \leq N-1,$$

$$\tilde{\mu}_{r-1,N} = \frac{1}{2}(2aN + br), \quad 2 \leq r \leq N,$$

the process $\zeta_N(t)$ conforms to the process $|X_N(t)|,\ t \geq 0$, associated with *CFP*'s(N) given by (2.10), (2.11). Finally, it is appropriate to note that after interchanging the roles of "0"-s and "1"-s, the spin system with the rates $\tilde{\lambda}_r,\ \tilde{\mu}_{r-1}$ as above, is known (for N fixed) as a contact process.

(ii) It follows from Proposition 1 that the class of *CFP*'s(N) that induce time homogeneous processes $|X_N(t)|,\ t \geq 0$ includes processes of pure coagulation ($\phi(1,1) = 0$ in (2.11)) and processes of pure fragmentation ($a = b = 0$ in (2.10)). Also observe that the time homogeneity requirement determines uniquely the form of rates of single coagulations, while it leaves a certain freedom in the choice of rates of single fragmentations.

As far as we know, there are no explicit solutions, i.e. explicit formulae for transition probabilities $\mathbb{P}(|X_N(t)| = r),\ t \geq 0,\ 1 \leq r \leq N$, for birth-death processes with the rates given by (2.17), when $a, b > 0,\ \psi(1,1) > 0$ and the initial distribution is concentrated on some state $\zeta \in \Omega_N$. The problem here is that the birth and death rates in (2.17) are polynomials in r of different degrees, which are 1 and 2 respectively. A survey of solvable birth-death processes with polynomial rates is given in [22].

We will see in the next subsection that under the above condition of time homogeneity of $|X_N(t)|,\ t \geq 0$ and certain initial distributions ρ, the corresponding *CFP*'s(N) are solvable.

2.2 Solvable *CFP*'s

Let a *CFP*(N) considered start from an initial distribution ρ on Ω_N, with projections ρ_r on the sets $\Omega_{N,r},\ r = 1, \dots, N$:

$$\rho_r(\eta) := \begin{cases} \rho(\eta \mid |\eta| = r), & \eta \in \Omega_{N,r}, \quad \text{if} \quad \rho(\Omega_{N,r}) := \rho(|\eta| = r) > 0, \\ 0, \quad \eta \in \Omega_{N,r}\ , & \qquad\qquad\ \ \text{if} \quad \rho(\Omega_{N,r}) = 0. \end{cases} \tag{2.18}$$

It is in order to note that the set of all distributions ρ on Ω_N with given projections ρ_r, is

$$\left\{ \rho : \rho(\eta) = \rho_r(\eta)\rho(\Omega_{N,r}),\ \eta \in \Omega_{N,r},\ \sum_{r=1}^{N} \rho(\Omega_{N,r}) = 1,\ \rho(\Omega_{N,r}) \geq 0,\ r = 1, \dots, N \right\}, \tag{2.19}$$

i.e. the projections $\rho_r,\ r = 1, \dots, N$ define the associated distribution ρ up to the factors $\rho(\Omega_{N,r}),\ r = 1, \dots, N$ in (2.19).

Accordingly, we write

$$p_\rho(\eta;t) = \mathbb{P}\Big(X_N^{(\rho)}(t)=\eta \mid |X_N^{(\rho)}(t)|=|\eta|\Big)\,\mathbb{P}\Big(|X_N^{(\rho)}(t)|=|\eta|\Big)$$
$$:= Q(\eta,\rho;t)\, b(|\eta|,\rho;t), \quad \eta\in\Omega_N,\ t\geq 0, \tag{2.20}$$

where $Q(\eta,\rho;t)$ and $b(|\eta|,\rho;t)$ denote respectively the first and the second factors in the RHS of the first line, while the conditional probability Q obeys the initial conditions

$$Q(\eta,\rho;0)=\rho_r(\eta), \quad \eta\in\Omega_{N,r},\ r=1,\dots,N, \tag{2.21}$$

for any ρ on Ω_N, which follow from the definitions of Q and ρ_r.

We will be interested in *CFP*'s possessing a conditional probability Q not depending of time under certain initial distributions ρ. If this is the case, it follows from (2.21) that

$$Q(\eta,\rho;t)=\rho_r(\eta), \quad \eta\in\Omega_{N,r},\ r=1,\dots,N,\ t\geq 0. \tag{2.22}$$

Obviously, time independence (2.22) holds for any $CFP(N)$ starting from its stationary distribution. In view of this, we adopt the following convention.

Definition 1 A $CFP(N)$ possesses a time independent conditional probability Q if (2.22) holds for certain projections ρ_r on $\Omega_{N,r},\ r=1,\dots,N$ and all initial distributions ρ on Ω_N from the associated set (2.19).

Next we write the Kolmogorov system for a birth and death process $|X_N^{(\rho)}(t)|,\ t\geq 0$ with rates $\lambda_{r,N}(t;\rho),\ \mu_{r,N}(t;\rho)$:

$$\dot b(r,\rho;t) = -b(r,\rho;t)\big(\lambda_{r,N}(t;\rho)+\mu_{r,N}(t;\rho)\big)+b(r+1,\rho;t)\mu_{r+1,N}(t;\rho)$$
$$+\,b(r-1,\rho;t)\lambda_{r-1,N}(t;\rho), \quad r=1,\dots,N, \tag{2.23}$$

where $b(0,\rho;t)=b(N+1,\rho;t)=0,\ t\geq 0$.

The following assertion is crucial for our study.

Proposition 2

The following two conditions (i) *and* (ii) *are equivalent.*

(i) *A $CFP(N)$ possesses a conditional probability Q independent of time $t\geq 0$;*
(ii) *The birth and death process $|X_N(t)|,\ t\geq 0$ is time homogeneous.*

Moreover, the projections $\rho_r,\ r=1,\dots,N$ defining by (2.22) *the time independent conditional probability Q are the unique solution of the two systems of equations*:

$$\mu_{r+1,N}\rho_r(\eta) = \sum_{\zeta\in\ \Omega_{N,r+1}:\ \zeta\sim\eta} \rho_{r+1}(\zeta)K(\zeta\to\eta), \quad \eta\in\Omega_{N,r},\ r=1,\dots,N-1, \tag{2.24}$$

$$\lambda_{r,N}\rho_{r+1}(\zeta) = \sum_{\eta\in\ \Omega_{N,r}:\ \eta\sim\zeta} \rho_r(\eta)F(\eta\to\zeta), \quad \zeta\in\ \Omega_{N,r+1},\ r=1,\dots,N-1, \tag{2.25}$$

where the rates of state transitions F and K are given by (1.1), (2.10), (2.11), *while the rates of birth and death are as in* (2.17).

Proof We substitute (2.20) in the Kolmogorov system (2.1) to obtain

$$\begin{aligned}
&\dot{b}(r,\rho;t)Q(\eta,\rho;t)+\dot{Q}(\eta,\rho;t)b(r,\rho;t)\\
&\quad=-Q(\eta,\rho;t)b(r,\rho;t)\Bigg(\sum_{\zeta\in\,\Omega_{N,r-1}:\,\zeta\sim\eta}K(\eta\to\zeta)+\sum_{\zeta\in\,\Omega_{N,r+1}:\,\zeta\sim\eta}F(\eta\to\zeta)\Bigg)\\
&\qquad+b(r+1,\rho;t)\sum_{\zeta\in\Omega_{N,r+1}:\,\zeta\sim\eta}Q(\zeta,\rho;t)K(\zeta\to\eta)\\
&\qquad+b(r-1,\rho;t)\sum_{\zeta\in\Omega_{N,r-1}:\,\zeta\sim\eta}Q(\zeta,\rho;t)F(\zeta\to\eta),\\
&\quad\eta\in\Omega_{N,r},\ r=1,\dots,N,\ t\ge 0,\\
&\Omega_{0,N}=\Omega_{N,N+1}=\emptyset,\qquad b(0,\rho;t)=b(N+1,\rho;t)=0,\quad t\ge 0.
\end{aligned}\tag{2.26}$$

We firstly prove the implication (ii) $\Rightarrow$ (i). We substitute in the LHS of (2.26) the expression for the derivative $\dot{b}(r,\rho;t)$ from (2.23), assuming that the process $|X_N(t)|,\ t\ge 0$ is time homogeneous and that the initial distribution is ρ. Then, by virtue of Lemma 1, the system (2.26) becomes

$$\begin{aligned}
&Q(\eta,\rho;t)\Big(b(r+1,\rho;t)\mu_{r+1,N}+b(r-1,\rho;t)\lambda_{r-1,N}\Big)+\dot{Q}(\eta,\rho;t)b(r,\rho;t)\\
&\quad=b(r+1,\rho;t)\sum_{\zeta\in\Omega_{N,r+1}:\,\zeta\sim\eta}Q(\zeta,\rho;t)K(\zeta\to\eta)\\
&\qquad+b(r-1,\rho;t)\sum_{\zeta\in\Omega_{N,r-1}:\,\zeta\sim\eta}Q(\zeta,\rho;t)F(\zeta\to\eta),\\
&\quad\eta\in\Omega_{N,r},\ r=1,\dots,N,\ t\ge 0,\\
&\Omega_{0,N}=\Omega_{N,N+1}=\emptyset,\qquad b(0,\rho;t)=b(N+1,\rho;t)=0,\quad t\ge 0,\ \text{for all } \rho \text{ on } \Omega_N,
\end{aligned}\tag{2.27}$$

where by the assumption made, the rates $K(\zeta\to\eta)$ and $F(\eta\to\zeta)$ of state transitions are implied by (2.10), (2.11) respectively and the birth and death rates are as in Corollary 1. Given an initial distribution ρ, constants a, b and fragmentation rates $\phi(i,j)$ obeying (2.11), a finite Kolmogorov system (2.27) has a unique solution Q, provided $a^2+b^2+\phi(1,1)>0$. In particular, (2.27) is satisfied by the time-independent Q, such that (2.22) holds for all initial distributions ρ with the projections $\rho_r,\ r=1,\dots,N$ that obey (2.24), (2.25).

To prove the implication (i) $\Rightarrow$ (ii), we observe that by virtue of (2.7), the condition (i) implies that the birth and death rates do not depend on $t\ge 0$. By our remark after (2.3) this leads to the conclusion that the rates do not depend on ρ either.

Next, we set $t=0$ in (2.27) to derive by virtue of Definition 1 and the time homogeneity of $|X_N(t)|,\ t\ge 0$, that the projections $\rho_r,\ r=1,\dots,N$ should obey (2.24) and (2.25).

It is left to show the existence and uniqueness of the solution $\rho_r,\ r=1,\dots,N$ for the system of (2.24), (2.25), where the rates of state transitions are induced by ψ and ϕ as in (2.10), (2.11). Recalling Lemma 1, we treat the ratios

$$P_C(\zeta\to\eta):=\frac{K(\zeta\to\eta)}{\mu_{r+1,N}},\quad \zeta\in\Omega_{N,r+1},\eta\in\Omega_{N,r},\ \zeta\sim\eta,\ r=1,\dots,N-1$$

as the one-step transition probabilities of a discrete time nearest-neighbor "coagulation" random walk on the set of partitions Ω_N. Then $\rho_r(\eta),\ \eta\in\Omega_{N,r}$, in (2.24) can be interpreted as the probability that the random walk starting at $\eta^* = (N, 0, \dots, 0) \in \Omega_{N,N}$ reaches a given state $\eta \in \Omega_{N,r}$ at the $(N-r)$-th step, so that $\zeta^* = (0, \dots, 1)$ is the absorbing state. In a similar manner, we consider the nearest neighbor "fragmentation" random walk on Ω_N with the transition probabilities

$$P_F(\eta \to \zeta) := \frac{F(\eta \to \zeta)}{\lambda_{r,N}}, \quad \eta \in \Omega_{N,r},\ \zeta \in \Omega_{N,r+1},\ \zeta \sim \eta,\ r = 1, \dots, N-1,$$

that starts at $\zeta^* = (0, \dots, 0, 1) \in \Omega_{N,1}$. In this case, $\rho_r(\eta),\ \eta \in \Omega_{N,r}$ in the set of equations (2.25) is the probability that the "fragmentation" random walk reaches a given state $\eta \in \Omega_{N,r}$ at the $(r-1)$-th step, $\eta^* = (N, \dots, 0)$ being the absorbing state. Clearly, each one of the two systems has a unique solution $\rho_r,\ r = 1, \dots, N$ whenever $a^2 + b^2 > 0$ in the first case and $\phi(1,1) > 0$ in the second case.

We demonstrate that when $(a^2 + b^2)\phi(1,1) > 0$ (=both coagulation and fragmentation hold), the two systems of equations have the same solution if and only if the transition probabilities P_C and P_F are related in the following way. Let $\rho_r,\ r = 1, \dots, N$ be the probabilities corresponding to the "coagulation" random walk, under some fixed $a, b : a^2 + b^2 > 0$ in (2.10). Then (2.25) for the "fragmentation" random walk have the same solution $\rho_r,\ r = 1, \dots, N$ if and only if

$$\rho_r(\eta) P_F(\eta \to \zeta) = \rho_{r+1}(\zeta) P_C(\zeta \to \eta), \quad \eta \in \Omega_{N,r},\ \zeta \in \Omega_{N,r+1},\ \eta \sim \zeta. \tag{2.28}$$

The sufficiency of (2.28) is seen immediately, while the necessity can be derived from the following general reasoning, based on the observation that each one of (2.24) and (2.25) is time reversal of the other one. Let $\rho_r,\ r = 1, \dots, N$ be the common solution of (2.24) and (2.25). Then from (2.24), applied for $r = N-2$, we conclude that under given ρ_{N-1} and ρ_{N-2}, the values a, b in (2.10) are uniquely determined. Hence, $\rho_r,\ r = 1, \dots, N$ uniquely determine all probabilities P_C in (2.24) induced by (2.10). If now some $\tilde{P}_F$ satisfies (2.25) under the above $\rho_r,\ r = 1, \dots, N$, then (2.24) should be satisfied by

$$\tilde{P}_C(\zeta \to \eta) = \frac{\rho_r(\eta) \tilde{P}_F(\eta \to \zeta)}{\rho_{r+1}(\zeta)}.$$

The aforementioned uniqueness of the probability P_C proves the claim. (In the discussion following the proof we find explicitly the solution $\rho_r,\ r = 1, \dots, N$ and demonstrate that the rates of single fragmentations derived from (2.28) satisfy the condition (2.11).) □

Our next purpose will be to find explicitly the solution $\rho_r(\eta),\ \eta \in \Omega_{N,r},\ r = 1, \dots, N$ of (2.24), (2.25), in the case when the time homogeneous process $|X_N(t)|,\ t \ge 0$ is given by (2.17). Since the sets $\Omega_{N,1}, \Omega_{N,N}$ are singletons, it follows from the definition of the conditional probability ρ_r that $\rho_N(\eta^*) = 1,\ \rho_1(\zeta^*) = 1$. The following two cases should be broadly distinguished.

Case 1: Non zero coagulation, i.e. $a^2 + b^2 > 0$.

Following the illuminating idea of Hendriks et al. [13], we will seek the probabilities ρ_r in question in the form

$$\rho_r(\eta) = \rho_{N,r}(\eta) = \left(B_{N,r}\right)^{-1} \frac{a_1^{n_1} a_2^{n_2} \dots a_N^{n_N}}{n_1! n_2! \dots n_N!},$$
$$\eta = (n_1, \dots, n_N) \in \Omega_{N,r},\ r = 1, \dots, N, \tag{2.29}$$

where $B_{N,r}$ is the normalizing constant (=partition function) known as the (N,r) partial Bell polynomial (see e.g. [4, 21]) induced by the sequence of weights $\{a_k\}_1^\infty$ that do not depend neither on N nor r. It follows from (2.29) that for given $\eta = (n_1, \dots, n_N) \in \Omega_{N,r}$, such that $n_{i+j} > 0$ for some $2 \le i + j \le N$, and $\zeta = \eta_{(i,j)} \in \Omega_{N,r+1}$,

$$\frac{\rho_{r+1}(\zeta)}{\rho_r(\eta)} = \left(\frac{B_{N,r}}{B_{N,r+1}}\right) \begin{cases} (\frac{a_i a_j}{a_{i+j}})\ (\frac{n_{i+j}}{(n_i+1)(n_j+1)}), & \text{if } i \neq j, \\ (\frac{a_i^2}{a_{2i}})\ (\frac{n_{2i}}{(n_i+1)(n_i+2)}), & \text{if } i = j. \end{cases} \tag{2.30}$$

Hence, setting, in accordance with Proposition 1 and (1.1),

$$K(\eta_{(i,j)} \to \eta) = \begin{cases} (a(i+j)+b)(n_i+1)(n_j+1), & \text{if } i \neq j, \\ (2ia+b)\frac{(n_i+1)(n_i+2)}{2}, & \text{otherwise,} \end{cases} \tag{2.31}$$

where $a^2 + b^2 > 0$, the system (2.24) conforms to

$$\mu_{r+1,N} = \left(\frac{B_{N,r}}{B_{N,r+1}}\right) \sum_{k=2}^{N} \frac{(ak+b)\sum_{i+j=k} a_i a_j}{2a_k}\, n_k,$$

$$(n_1, \dots, n_N) \in \Omega_{N,r},\ r = 1, \dots, N-1. \tag{2.32}$$

Since the RHS of (2.32) should not depend on $\eta \in \Omega_{N,r}$ the equations are solved by the weights defined recursively by

$$a_k = \frac{(ak+b)\sum_{i+j=k} a_i a_j}{2(k-1)}, \quad k \ge 2,\ a_1 = 1. \tag{2.33}$$

This is just the solution obtained, by quite different considerations, in [13] (see (18) there), for pure coagulation processes.

Continuing (2.32), we get

$$\mu_{r+1,N} = \left(\frac{B_{N,r}}{B_{N,r+1}}\right) \sum_{k=2}^{N} (k-1) n_k, \tag{2.34}$$

$$(n_1, \dots, n_N) \in \Omega_{N,r},\ r = 1, \dots, N-1, \tag{2.35}$$

which leads to the following relation between the constants $\mu_{r+1,N},\ B_{N,r},\ B_{N,r+1}$ induced by the weights (2.33):

$$\mu_{r+1,N} = (N-r)\left(\frac{B_{N,r}}{B_{N,r+1}}\right), \quad r = 1, \dots, N-1. \tag{2.36}$$

Taking into account that $B_{N,N} = \frac{a_1^N}{N!} = (N!)^{-1}$, we get the explicit expressions for the Bell polynomials in the case considered:

$$B_{N,r} = \frac{\prod_{l=r+1}^{N} \mu_{l,N}}{N!(N-r)!}, \quad r = 1, \dots, N-1, \tag{2.37}$$

where $\mu_{l,N}$ as in (2.17). Remarkably, the expression (2.37) for the Bell polynomials enables us to find explicitly the weights $a_k,\ k \ge 1$, without solving the recurrence relation (2.33). In fact, by (2.29),

$$a_N = B_{N,1} = \frac{\prod_{r=2}^{N} \mu_{r,N}}{N!(N-1)!}, \quad N \geq 2, \tag{2.38}$$

which can be written as

$$a_1 = 1, \qquad a_k = \frac{\prod_{r=2}^{k}(ka + \frac{br}{2})}{k!}, \quad k \geq 2,\ 2a + b > 0,\ a \geq 0. \tag{2.39}$$

Remark 2 The recurrence relation (2.33) can be viewed as a modification of the classic convolution formula,

$$a_k = \frac{1}{2} \sum_{i+j=k} a_i a_j, \quad k = 2, \ldots,\ a_1 = 1,$$

which determines the Catalan numbers (see e.g. [16]). It is interesting to find the generating function $g(x) = \sum_{k=1}^{\infty} a_k x^k$ for the sequence of weights $\{a_k\}_1^{\infty}$, defined by (2.33). Setting $y(x) = \frac{g(x)}{x}$, it follows from (2.33) that the function y obeys the differential equation

$$y'(1 - axy) = y^2 a_2, \quad a_2 = \frac{2a + b}{2},\ y(0) = a_1 = 1,$$

which implicit solution is given by

$$y(x) = \left(1 + \frac{b}{2}xy\right)^{\frac{2a+b}{b}}, \quad b > 0.$$

We now recover the fragmentation rates given by (2.28) in the case of coagulation rates (2.31). By (2.30), (2.36) and (2.17) we have

$$F(\eta \to \zeta) = \phi(1,1) \begin{cases} (\frac{a_i a_j\ n_{i+j}}{a_{i+j}})\ (a(i+j) + b), & \text{if } i \neq j, \\ (\frac{a_i^2\ n_{2i}}{2a_{2i}})\ (2ai + b), & \text{if } i = j \end{cases} \tag{2.40}$$

for $\eta = (n_1, \ldots, n_N) \in \Omega_{N,r}$, such that $n_{i+j} > 0$ for some $2 \leq i + j \leq N$, and $\zeta = \eta_{(i,j)} \in \Omega_{N,r+1}$. Note that by (2.33), the rates of single fragmentations induced by (2.40) satisfy the condition (2.11). This latter condition appears to have a physical meaning in the context of *CFP*'s describing polymerization (see [23]).

Also note that in the case considered, $\rho_r(\eta) > 0,\ \eta \in \Omega_{N,r},\ r = 1, \ldots, N$ and that the rates of single transitions are determined by (2.40) up to the constants a, b and $\phi(1,1)$.

Case 2: Pure fragmentation.

It is clear from the preceding discussion that under the fragmentation rates of the form (2.40) (with a, b that are not related to coagulation rates) the solution ρ_r of (2.25) is given by (2.29). However, in contrast to the case of pure coagulation, Proposition 1 leaves freedom for the choice of rates of single fragmentations obeying (2.11). In view of this, the probabilities ρ_r solving (2.25) will depend on a particular choice of the above rates, so that ρ_r, will be of Hendriks et al. form (2.29) if and only if the rates of single fragmentations are induced by (2.40). This is illustrated by the toy example below. (We recall that under all above choices of rates of single fragmentations, the rates of the induced pure birth process remain the same: $\lambda_r = \phi(1,1)(N - r),\ r = 1, \ldots, N$.)

Example 2 Let

$$\phi(i,j)=\phi(1,1)\begin{cases}(i+j-1), & \text{if } i=1 \text{ or } j=1,\\ 0, & \text{otherwise,}\end{cases} \tag{2.41}$$

so that (2.11) holds. The corresponding fragmentation random walk is in effect a deterministic chain on N states $\zeta_1,\dots,\zeta_r,\dots,\zeta_N$, such that

$$\zeta_r=(r-1,0,\dots,0,\overbrace{1}^{N-r+1},0,\dots,0)\in\Omega_{N,r},\quad 1\le r\le N-1,$$
$$\zeta_N=(N,0,\dots,0).$$

Consequently, (2.25) implies $\rho_r(\eta)=\mathbf{1}_{\zeta_r}(\eta),\ \eta\in\Omega_{N,r}$, which is not of the form (2.29).

The preceding discussion is summarized in our main theorem.

Theorem 1 (Solvable *CFP*'s) *Mean-field CFP's $X_N^{(\rho)}(t),\ t\ge 0$ with rates of single transitions* (2.10) *and* (2.11) *and initial distributions ρ on Ω_N, s.t. ρ_r on $\Omega_{N,r},\ r=1,\dots,N$ satisfy* (2.24), (2.25), *have time dynamics given by*

$$p_\rho(\eta;t)=\rho_r(\eta)\,b(r,\rho;t),\quad \eta\in\Omega_{N,r},\ r=1,\dots,N,\ t\ge 0, \tag{2.42}$$

where $b(r,\rho;t)$ are transition probabilities of the associated time homogeneous birth and death process $|X_N^{(\rho)}(t)|,\ t\ge 0$ with rates (2.17). *In particular, if rates of single coagulations are positive, then ρ_r is given by* (2.29), (2.37) *and* (2.39), *while in the case of pure fragmentation $\rho_r,\ r=1,\dots,N$ satisfying* (2.25) *are as before, if and only if* (2.40) *holds.*

Note that under $b=0$ in (2.17), the corresponding birth and death process is known as the Ehrenfest process (=urn model).

Remark 3 (Initial distributions ρ) *CFP*'s(N) with single transitions (2.10), (2.11) but with initial distributions ρ that do not obey the equations (2.24), (2.25) in Proposition 2 are not solvable, since in this case the conditional probability Q depends on $t\ge 0$ and ρ, though the processes $|X_N^{(\rho)}(t)|,\ t\ge 0$ are time homogeneous.

Remark 4 (Transition rule for Gibbs fragmentation) We now explain that the probabilities P_F induced by the positive fragmentation rates (2.40) define the following simple rule of a state transition via a fragmentation from $\eta\in\Omega_{N,r}$ to $\eta_{(i,j)}\in\Omega_{N,r+1}$. By (2.33),

$$P_F(\eta\to\eta_{(i,j)})=\frac{(i+j-1)n_{i+j}}{N-r}\begin{cases}a_ia_j\,(\frac12\sum_{l+m=\,i+j}\ a_la_m)^{-1}, & \text{if } i\ne j,\\ \frac{a_i^2}{2}(\frac12\sum_{l+m=2i}\ a_la_m)^{-1}, & \text{if } i=j.\end{cases} \tag{2.43}$$

Under a given $\eta=(n_1,\dots,n_N)\in\Omega_{N,r}$, the first factor in the above expression is the probability that a component of size $i+j\ge 2$ is selected to fragmentate, while the second factor specifies the probability that, conditioned on the first event, the selected component splits into two components of given sizes i and j. As a result, (2.43) conforms to a transition procedure postulated in [4], in which the first and the second factors are called the linear selection rule and the Gibbs splitting rule respectively. Theorem 1 and Proposition 2 say that the mean-field transition mechanism (2.43) is forced by the requirement that the process $|X_N(t)|,\ t\ge 0$ is time homogeneous and the rates of single coagulations are positive.

A Historical Note This note concerns exclusively the research on solvable *CFP*'s. Time evolution of the stochastic model of pure coagulation was formulated by Marcus [19] who was also apparently the first to reveal the relationship between Kolmogorov equations and its deterministic analog presented by Smoluchowski equations. Solutions to Smoluchowski equations for pure coagulation with kernels K induced by $\phi(i,j)\equiv const$, $\phi(i,j)=i+j$ and $\phi(i,j)=ij$ were obtained long ago by researchers in the field of colloid aerosol chemistry (for references see [1, 19]). Lushnikov [18] derived explicit formulae for the expected numbers $\mathbb{E}n_j(t),\ t\geq 0,\ j=1,\dots,N$ for the process $X_N^{(\zeta)}(t),\ t\geq 0$ of pure coagulation with $\phi(i,j)=i+j,\ i,j\geq 1$, with the help of the generating function for transition probabilities $p_\zeta(\eta;t),\ t\geq 0,\ \ \zeta,\eta\in\Omega_N$. The aforementioned stochastic model is known as the Marcus-Lushnikov process. In [18], treating Smoluchowski equations as an approximation to Kolmogorov ones, Lushnikov proved the important fact that the solution to Smoluchowski coagulation equations with a general coagulation kernel, can be presented as a mixture of Poisson distributions with time dependent parameters. (Note that these parameters were found explicitly for the Marcus -Lushnikov model only.) A further important contribution was made by Hendriks, Spouge, Eibl and Schreckenberg [13] who found explicitly the transition probabilities $p_\zeta(\eta;t),\ t\geq 0,\ \zeta,\eta\in\Omega_N$ for a more general Marcus-Lushnikov model with $\phi(i,j)$ as in (2.10). This result, proven via a combinatorial argument, is based on the representation (2.20) with time independent conditional probability Q.

2.3 Discussion of the Main Result

- *Steady state distribution.* Firstly, consider solvable *CFP's* with nonzero rates of single coagulations and fragmentations. By (1.1), (2.31), (2.40) and Theorem 1, the implied rates ψ,ϕ of single coagulations and fragmentations respectively, are

$$\psi(i,j)=a(i+j)+b,\quad \phi(i,j)=\phi(1,1)\frac{a_ia_j}{a_{i+j}}\Big(a(i+j)+b\Big),\quad i,j\geq 1,\ \phi(1,1)>0,$$

where $a_j,\ \ j\geq 1$ are given by (2.39). Thus, the ratio of the above rates is equal to

$$\frac{\psi(i,j)}{\phi(i,j)}=\frac{a_ia_j}{\phi(1,1)a_{i+j}},\quad i,j\geq 1. \tag{2.44}$$

Setting in (2.44) $\tilde{a}_i=\frac{a_i}{\phi(1,1)}$, gives

$$\frac{\psi(i,j)}{\phi(i,j)}=\frac{\tilde{a}_i\tilde{a}_j}{\tilde{a}_{i+j}},\quad i,j\geq 1,$$

which shows that the criteria of reversibility of mean- field *CFP*'s (see [5]) is fulfilled. Moreover, by Theorem 1, the above process is the only reversible process, within the class of solvable mean-field *CFP*'s. By virtue of (2.42), the invariant measure ν_N of the process considered is

$$\nu_N(\eta)=b(r)\rho_r(\eta),\quad \eta\in\Omega_{N,r},\ r=1,\dots,N, \tag{2.45}$$

where $b(r)=\lim_{t\to\infty}b(r,\rho;t)$ is the invariant measure of the associated ergodic birth and death process (see for reference [2]). The probability measures ρ_r on $\Omega_{N,r},\ r=1,\dots,N$ defined by (2.29), (2.37), (2.39), belong to the class of multiplicative measures (=Gibbs distributions) which play also a role in the theory of random combinatorial structures (see [6–8, 21, 24]). The explicit expression for the measure ν_N is obtained in a straightforward way from the known form (see [5]) of the invariant measure of a reversible *CFP* with rates (2.44):

$$\nu_N(\eta) = (c_N)^{-1}\Big(\frac{\tilde{a}_1^{n_1}\tilde{a}_2^{n_2}\dots\tilde{a}_N^{n_N}}{n_1!n_2!\dots n_N!}\Big), \qquad \eta\in\Omega_N, \tag{2.46}$$

where c_N is the partition function of the measure ν_N. Next, we embark on an analysis of the asymptotic behaviour of the measure ν_N, as $N\to\infty$. For this purpose we need to know the asymptotics of the weights $\{a_k\}_1^\infty$. By (2.39),

$$a_k = \begin{cases} \frac{(\frac{b}{2})^{k-1}}{k!}\prod_{r=2}^k(\frac{2a}{b}k+r) = (\frac{b}{2})^{k-1}(k!(\frac{2a}{b}k)(\frac{2a}{b}k+1))^{-1}(\frac{2a}{b}k)_{k+1}, & \text{if } b\neq 0,\\ \frac{a^{k-1}k^{k-1}}{k!}, & \text{if } b=0, \end{cases} \tag{2.47}$$

where $k\geq 1$ and $(z)_n := z(z+1)\dots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$ is the Pochhammer symbol. Applying the Stirling's approximation, gives, as $k\to\infty$,

$$a_k \sim \begin{cases} C_1C_2^k\, k^{-\frac{3}{2}}, & \text{if } ab>0,\\ (\frac{b}{2})^{k-1}, & \text{if } a=0,\ b>0,\\ C_3C_4^k\, k^{-\frac{3}{2}}, & \text{if } b=0,\ a>0, \end{cases} \tag{2.48}$$

where $C_1 = C_1(a,b),\ C_2 = C_2(a,b),\ C_3 = C_3(a),\ C_4 = C_4(a)$ are positive constants. The measure ν_N in (2.46) is invariant under the transformations of the weights $a_k\to C^ka_k$, with any constant $C>0$. Thus, the asymptotic behaviour of the measure ν_N considered is identical to the one with the weights

$$a'_k \sim \begin{cases} C_1\, k^{-\frac{3}{2}}, & \text{if } ab>0,\\ const, & \text{if } a=0,\ b>0,\\ C_3\, k^{-\frac{3}{2}}, & \text{if } b=0,\ a>0, \end{cases} \tag{2.49}$$

as $k\to\infty$. In accordance with the classification suggested in [3] for multiplicative measures ν_N with regularly varying weights $a_k\sim k^\alpha,\ k\to\infty$, the measure ν_N considered belongs to the convergent class ($\alpha<-1$) in the first and the third cases in (2.49), while in the second case in (2.49) it belongs to the expansive class ($\alpha>-1$). It was shown in [3] that the convergent class of ν_N exhibits a strong gelation, as $N\to\infty$: with a positive probability all groups cluster in one huge component of size close to N. In contrast to this, (see [8]), the expansive measures ν_N have, with probability 1, as $N\to\infty$, a threshold value $N^{\frac{1}{\alpha+2}}$ for the size of the largest group in the associated random partition. In the context of the *CFP* considered the above crucial difference is easily explained by noting that the first and the third cases in (2.49) correspond to a "strong" coagulation, while the second case corresponds to a coagulation with a constant rate.

Clearly, pure coagulation and pure fragmentation processes $X_N(t),\ t\geq 0$ have the absorbing states $\eta^* = (0,\dots,1)$ and $\zeta^* = (N,0,\dots,0)$ respectively.

In the conclusion consider a non-solvable *CFP*(N) as in Remark 3. In view of the ergodicity of this process, its invariant measure will be identical to the one of a solvable *CFP*, starting from any distribution ρ on Ω_N with the gibbsian projections ρ_r on $\Omega_{N,r}$, $r=1,\dots,N$, given by (2.29).

- *CFP's on set partitions*. These are processes with values in the space $\Omega_{[N]}$ of partitions of the set $[N]=\{1,2,\dots,N\}$ (=set partitions). From the physical point of view, this means that, in the setting of this paper, the N particles are labelled, so that clusters forming a state of the process are subsets of the set $[N]$. $\Omega_{[N]}$-valued *CFP*'s are a generalization of Kingman's coalescent that provided a mathematical framework for a variety of genetic models, in particular the Ewens sampling formula. Kingman's theory, which is surveyed in [21], is based on the theory of exchangeable partitions. The development of Kingman's coalescent by Pitman [21] and his colleagues lead to Gibbs partitions as distributions of $\Omega_{[N]}$-valued irreversible processes of pure fragmentation or pure coagulation. Formally, the linkage between Ω_N-valued and $\Omega_{[N]}$-valued *CFP*'s is expressed via a simple combinatorial formula and it is discussed in [4, 7, 21]. Among *CFP*'s on $\Omega_{[N]}$, Gibbs fragmentation processes introduced in [4] by N. Berestycki and Pitman play a central role. These processes are defined as time homogeneous Markov chains $\Pi(t) \in \Omega_{[N]},\ t \geq 0$ of pure fragmentation, such that the conditional distribution of $\Pi(t)$ given the number of blocks of the random set partition $\Pi(t)$ is a microcanonical Gibbs distribution not depending of $t \geq 0$. In terms of CFP's on Ω_N, the above conditional distributions are just the distributions (2.29) on $\Omega_{N,r},\ 1 \leq r \leq N$. Correspondingly, the time reversal of the above process is called Gibbs coagulation. In [4] the authors posed a problem of characterization of the weights (in their notation) $\omega_k := a_k k!$ for which there exist Gibbs fragmentation processes, and they proved that, under the assumption that the fragmentation rates are defined by recursive and selection rules (2.43), the unique Gibbs distribution is given by the weights (2.33). In [4, p. 393] it was conjectured that some other, more complicated splitting rules might be of interest. We will demonstrate (see Proposition 3 below) that the aforementioned characterization of weights is valid for a broad class of fragmentation rules, that includes the above one in [4].

The problem reduces (see Problem 2 in [4]) to the characterization of weights $a_k, k \geq 1$ (not depending of N) and transition probabilities of fragmentations P_F that satisfy (2.25):

$$\rho_{N,r+1}(\zeta) = \sum_{\eta \in\ \rho_{N,r}:\ \eta \sim \zeta} \rho_{N,r}(\eta) P_F(\eta \to \zeta), \quad \zeta \in\ \Omega_{N,r+1}, r = 1, \ldots, N-1, \tag{2.50}$$

when $\rho_{N,r}$ is a Gibbs measure (2.29) on $\Omega_{N,r}$. Regarding the probabilities P_F, we assume that they are of the following general form implied by the mean-field property:

$$P_F(\eta \to \eta_{(i,j)}) = \frac{n_{i+j}\phi(i,j)}{c(\eta)}, \quad \eta = (n_1, \ldots, n_N) \in \Omega_N, \tag{2.51}$$

where $\phi(i,j)$ is a symmetric nonnegative function not depending of N and $c(\eta) = \sum_{1 \leq i \leq j \leq N} n_{i+j}\phi(i,j)$ is the normalizing constant. Clearly, (2.43) is a particular case of (2.51).

For our subsequent considerations it is important to note that in (2.29) all weights $a_k,\ k \geq 1$ should be positive, due to the fact that $\frac{a_N}{B_{N,1}} = 1,\ N \geq 1$.

Proposition 3 *Under the assumption* (2.51), *Gibbs distributions* $\rho_{N,r},\ r = 1, \ldots, N$ *satisfy* (2.50) *if and only if the weights* a_k *in* (2.29) *are given by* (2.33) *and the rates* $\phi(i,j)$ *of single fragmentations are the same as in* (2.40).

Proof We assume that Gibbs distributions $\rho_{N,r},\ 1 \leq r \leq N$ satisfy (2.50). Treating (2.50) when $r = 1$ and

$$\zeta = (0, \ldots, \overset{i}{\overbrace{1}}, \ldots, 0, \overset{N-i}{\overbrace{1}}, 0, \ldots, 0) \in \Omega_{N,2}, \quad i = 1, \ldots, N-1,$$

gives

$$\left(\frac{B_{N,2}}{B_{N,1}}\right)\left(\frac{a_N}{a_i a_{N-i}}\right)\left(\frac{\phi(i,N-i)}{v_N}\right)=1, \tag{2.52}$$

if $N \neq 2i$ and

$$\left(\frac{B_{2i,2}}{B_{2i,1}}\right)\left(\frac{a_{2i}}{\frac{1}{2}a_i^2}\right)\left(\frac{\phi(i,i)}{v_{2i}}\right)=1,$$

if $N=2i$, where in both cases $v_k>0$ is defined as in (2.11). Since $B_{N,1}=a_N$, we have

$$0<\phi(i,N-i)=\frac{v_N}{B_{N,2}}\begin{cases} a_i a_{N-i}, & \text{if } N\neq 2i,\\ \frac{1}{2}a_i^2, & \text{if } N=2i.\end{cases} \tag{2.53}$$

Secondly, applying (2.50) for $r=2$ with

$$\zeta\in\Omega_{N,3}:\zeta(k_1)=\zeta(k_2)=\zeta(N-k_1-k_2)=1,$$

where $k_1,k_2,N-k_1-k_2$ are distinct positive integers, gives

$$1=\sum_{i=1}^{3}\frac{\rho_{N,2}(\eta_i)}{\rho_{N,3}(\zeta)}P_F(\eta_i\to\zeta), \tag{2.54}$$

where

$$\eta_1\in\Omega_{N,2}:\eta_1(k_1)=\eta_1(N-k_1)=1, \qquad \eta_2\in\Omega_{N,2}:\eta_2(k_2)=\eta_2(N-k_2)=1,$$
$$\eta_3\in\Omega_{N,2}:\eta_3(k_1+k_2)=\eta_3(N-k_1-k_2)=1$$

denote the three states from which it is possible to arrive, via one step fragmentation, at the above state $\zeta\in\Omega_{N,3}$. Substituting in (2.54) the expression (2.53) for ϕ and using (2.30), we obtain

$$1=\left(\frac{B_{N,3}}{B_{N,2}}\right)\left(\frac{a_{N-k_1}v_{N-k_1}}{B_{N-k_1,2}(v_{k_1}+v_{N-k_1})}+\frac{a_{N-k_2}v_{N-k_2}}{B_{N-k_2,2}(v_{k_2}+v_{N-k_2})}\right.$$
$$\left.+\frac{a_{k_1+k_2}v_{k_1+k_2}}{B_{k_1+k_2,2}(v_{k_1+k_2}+v_{N-k_1-k_2})}\right). \tag{2.55}$$

We set now for a given $N\geq 3$,

$$f_N(k):=\frac{a_k v_k}{B_{k,2}(v_k+v_{N-k})}, \qquad 2\leq k\leq N-1.$$

This allows us to rewrite (2.55) as

$$f_N(N-k_1)+f_N(N-k_2)+f_N(k_1+k_2)=C(N), \quad N\geq 3, \tag{2.56}$$

where $C=C(N)$ is a constant w.r.t. $k_1,k_2: N-k_1\geq 2,\ N-k_2\geq 2,\ k_1+k_2\geq 2$.

The solution of (2.56) is given by a linear function

$$f_N(k)=A_N k+B_N>0, \quad 2\leq k\leq N\geq 3$$

and the constant $C = 2A_N N + 3B_N,\ N \geq 3$, where the reals $A_N,\ B_N : A_N \geq 0,\ 2A_N + B_N > 0$. As a result, the following relation is derived

$$\frac{a_k v_k}{B_{k,2}(v_k + v_{N-k})} = A_N k + B_N, \quad 2 \leq k \leq N \geq 3. \tag{2.57}$$

We will show that (2.57) forces the weights $a_k,\ k \geq 2$ to satisfy (2.33). Let

$$0 \leq H_k := \limsup_{N\to\infty} (A_N k + B_N),$$

for any fixed $k \geq 2$. $H_k = \infty$ is impossible because $v_k > 0,\ k \geq 2$, by (2.53). Hence, $H_k \geq 0$ is finite for all $k \geq 2$, which implies

$$0 \leq A := \limsup_{N\to\infty} A_N < \infty, \qquad B := \limsup_{N\to\infty} B_N < \infty. \tag{2.58}$$

Recalling that $v_1 = 0$, we apply (2.57) with $N = k+1,\ k \geq 2$ and $N = 2k,\ k \geq 2$, to get

$$\frac{a_k}{B_{k,2}} = A_{k+1}k + B_{k+1}, \quad k \geq 2$$

and

$$\frac{a_k}{2B_{k,2}} = A_{2k}k + B_{2k}, \quad k \geq 2$$

respectively. In view of (2.58), the last two relations are in agreement if and only if $A = B = 0$, so that from (2.57) we recognize that

$$\lim_{N\to\infty} v_{N-k} = \infty, \quad k \geq 1.$$

Consequently, letting

$$z := \limsup_{N\to\infty} \frac{v_N}{v_{N-1}} \geq 1,$$

and denoting

$$\frac{a_k v_k}{B_{k,2}} = e_k > 0, \quad k \geq 2,$$

one obtains from (2.57)

$$e_k = \lim_{N\to\infty} v_{N-k}(A_N k + B_N) = z^{-k}(ak + b), \quad k \geq 1,$$

where

$$0 \leq \tilde{a} := \lim_{N\to\infty} v_N A_N < \infty, \qquad \tilde{b} =: \lim_{N\to\infty} v_N B_N < \infty. \tag{2.59}$$

Substituting the expression for e_k into (2.57) leads to the following relation

$$\frac{z^{-k}(\tilde{a}k + \tilde{b})}{v_k + v_{N-k}} = A_N k + B_N, \quad 1 \leq k \leq N-1,$$

which implies

$$\frac{z^{-k}(\tilde{a}k + \tilde{b}) + z^{-(N-k)}(\tilde{a}(N-k) + \tilde{b})}{v_k + v_{N-k}} = NA_N + 2B_N, \quad 1 \leq k \leq N-1. \tag{2.60}$$

Supposing $z>1$, we obtain, in view of (2.59),

$$z^{-k}(\tilde{a}k+\tilde{b}) = \lim_{N\to\infty} v_{N-k}(NA_N+2B_N)$$
$$= \begin{cases} \infty, & \text{if } \tilde{a}\neq 0, \\ z^{-k}(\lim_{N\to\infty}(Nv_NA_N)+2\tilde{b}), & \text{otherwise}, \end{cases} \quad k\geq 1. \tag{2.61}$$

In both cases this leads to contradiction, since in the case $\tilde{a}=0$, we should have $\tilde{b}>0$, by the definition of e_k. Hence, $z=1$. By (2.60), this means that for a fixed N, the sum v_k+v_{N-k} does not depend on k, so that v_k is linear in k, namely $v_k=\phi(1,1)(k-1)$, since $v_1=0,\ v_2=\phi(1,1)$. As a result, (2.57) becomes

$$\frac{a_k(k-1)}{B_{k,2}} = (N-2)A_Nk+(N-2)B_N = \frac{\tilde{a}k+\tilde{b}}{\phi(1,1)} := ak+b, \quad 2\leq k\leq N\geq 3,$$

since the LHS does not depend on N. Recalling now the definition of $B_{k,2}$, gives (2.33). □

Remark 5 (i) In [4], Berestycki and Pitman characterized the Gibbs solutions of (2.50) in the particular case of state transitions (2.43). Our solution (2.39) derived under a more restricted mean-field assumption (2.51) has the same form as in [4], but with less freedom on the constants a, b.

(ii) The weights $w_k=a_kk!$ in the form of a finite product of linear factors appear also as a solution of a quite different characterization problem. Gnedin and Pitman [9], extending Kerov's result [15], proved that an infinite sequence $\{\Pi_N\},\ N\geq 1$ of Gibbs random partitions of $[N]$ is exchangeable if and only if in (2.29) the weights, say $\tilde{w}_k=\tilde{a}_kk!$, are of the form

$$\tilde{w}_k=\prod_{l=1}^{k-1}\left(\tilde{b}l-\tilde{a}\right), \quad k\geq 2,\ w_1=1,\ \tilde{b}\geq 0,\ \tilde{a}\leq\tilde{b}.$$

In contrast to (2.39), the linear factors of $\tilde{w}_k$ do not depend k.

The first part of the following corollary gives an answer to Problem 3 in [4], in the class of mean-field *CFP*'s, while the second part recovers Proposition 1 in the above paper, in the aforementioned class of models.

Corollary 2 *For N large enough there do not exist mean-field Gibbs fragmentation processes on $\Omega_{[N]}$, with weights $w_k=(k-1)!, k\geq 1$ and $w_k\equiv const,\ k\geq 1$.*

Proof Recalling that $w_k=a_kk!,\ k\geq 1$, both assertions follow from (2.49) which says that the asymptotics, as $k\to\infty$ of the two types of weights in question are not of the form required in Proposition 3. □

Remark 6 In a recent paper [10] a non mean-field Gibbs fragmentation process with weights $w_k=(k-1)!,\ k\geq 1$ was constructed. The construction based on the Chinese restaurant model for simulation of uniform random permutation, results in a Gibbs fragmentation process with state transitions not obeying the mean field form (2.51).

- *Spectral gap.* By virtue of (2.42), the spectral gap of the solvable *CFP*'s considered is equal to the one of the Markov time homogeneous birth and death process $|X_N^{(\rho)}(t)|$,

$t \geq 0$ with the rates of birth $\lambda_r = \lambda_{r,N} = \phi(1,1)(N-r),\ r = 1, \dots, N-1$ and rates of death $\mu_r = \mu_{r,N} = \frac{r-1}{2}(2aN + rb),\ r = 1, \dots, N$. We shall employ Zeifman's method as described in [12], to find the spectral gap, say β_N, of the above birth and death process. Recalling that $\lambda_N = \mu_1 = 0$, consider the $N-1$ quantities

$$\alpha_r = \alpha_r(\vec{\delta}) := \lambda_r + \mu_{r+1} - \delta_{r+1}\lambda_{r+1} - \frac{\mu_r}{\delta_r}, \quad r = 1, \dots, N-1, \tag{2.62}$$

where $\vec{\delta} = \vec{\delta}_N = (\delta_r = \delta_{r,N} > 0,\ r = 2, \dots, N-1)$ is a vector of unknowns δ_r. The method states that

(i) For any vector $\vec{\delta}$,

$$\min\{\alpha_r,\ 1 \leq r \leq N-1\} \leq \beta_N \leq \max\{\alpha_r,\ 1 \leq r \leq N-1\}.$$

(ii) In the case of an ergodic birth and death process, there exists a unique vector $\vec{\delta}$, such that all $N-1$ quantities α_r are equal, so that their common value is equal to β_N.

In our case (2.62) conforms to

$$\begin{aligned}\alpha_r &= \phi(1,1)(N-r) + \frac{r}{2}\big(2aN + (r+1)b\big) - \phi(1,1)(N-r-1)\delta_{r+1} \\ &\quad - \frac{(r-1)(2aN+rb)}{2\delta_r}, \quad r = 1, \dots, N-1.\end{aligned} \tag{2.63}$$

Setting in (2.63) $\delta_r = 1,\ r = 2, \dots, N-1$, we obtain

$$\alpha_r = \phi(1,1) + aN + br, \quad r = 1, \dots, N-1,$$

from which the following two-sided bound for the β is derived:

$$\phi(1,1) + aN + b \leq \beta_N \leq \phi(1,1) + aN + b(N-1).$$

In particular, if $b = 0$, the preceding relation gives the exact value of the spectral gap $\beta_N = \phi(1,1) + aN$.

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